Realizability Problem for Constraint LTL

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Constraints and Constraint Systems

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- A constraint c is a relation either of the form $t_1 < t_2$ or $t_1 = t_2$.

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  where c is any constraint.
- $F\phi$ and $G\phi$ - derived operators as used in LTL.

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$$\begin{array}{cccccccc} x & 1 & 2 & 3 & 4 & 5 & \ldots & \\ & & & & & & & \models \phi \\ y & 2 & 4 & 6 & 8 & 10 & \ldots & \end{array}$$

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- The system is said to have a winning strategy if it is possible for the system to win every play of the game regardless of how the environment plays.
- Given a CLTL formula ϕ and an ownership of the variables, the realizability problem refers to the problem of checking whether the system has a winning strategy in the CLTL game.

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- The CLTL formula is such that the constraints in the formula cannot compare the values of the future-blind variables across different positions.
- In the single-sided game, the environment is constrained to only own future-blind variables. The system is free to own both lookahead and future-blind variables.

Single-sided CLTL games

- Consider $\phi_1 = G((x_e < x_s) \wedge (y_e < Xy_s))$. Here the current value of y_e can be compared with the value of y_s at the next position. So y_e is a lookahead variable and the CLTL game is not single-sided.

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- Now, let x_e, y_e, x_s be future-blind variables and y_s be a lookahead variable. Consider the single-sided CLTL game with winning condition $\phi_2 = G((x_e < x_s) \wedge (y_e < x_s) \wedge (y_s < Xy_s))$.

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- Undecidable for integers with linear order and equality
- 2EXPTIME-complete for single-sided games on integers with linear order and equality

Undecidability over $(\mathbb{Z}, <, =)$

- We prove undecidability of the realizability problem over $(\mathbb{Z}, <, =)$ using a standard game-theoretic technique where we reduce the reachability problem for 2-counter machines to the realizability problem.

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- CLTL formulas are used to control how the variable values are assigned so that the transitions of the counter machine are simulated faithfully.

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- In order to prove decidability, we use the technique of abstracting the concrete models using symbolic models.

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$$\begin{array}{cccccc} x & -1 & -2 & -3 & -4 & \dots \\ y & 1 & 2 & 3 & 4 & \dots \end{array} \models \phi$$

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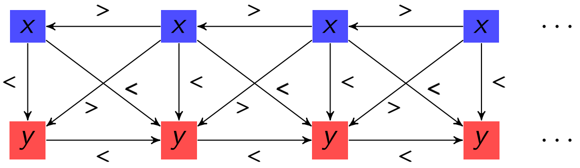

Symbolic Model Technique

- This technique has already been used to prove that the satisfiability problem for this logic is decidable.[1]

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- We can think of strategies in the CLTL game as an infinitely branching tree with labels from an infinite alphabet.
- We show that using the symbolic model technique, it is possible to reason using finitely branching trees with labels from a finite alphabet.

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Future Work

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- We believe that single-sided CLTL games over the natural numbers is also decidable. We plan to prove it by appropriately extending the techniques that we have used to prove decidability over the integers.

Thank You!